HW1

Exercise 1. Verify (and find the correct signs in) the cubic term of the Campbell–Hausdorff formula.

Exercise 2. Prove that a compact complex connected Lie group G must be abelian:

(a) Verify that the map $Ad : G \to \mathrm{Aut}(T_eG) \subset \mathrm{End}(T_eG)$ is holomorphic, and, therefore (by the maximum principle), constant.

(b) Deduce that if Ψ_g is conjugation by g, then $d\Psi_g$ is the identity, so $\Psi_g(\exp(X)) = \exp(d\Psi_g(X)) = \exp(X)$ for all $X \in T_eG$, which implies that G is abelian.

(c) Show that the exponential map from T_eG to G is surjective, with the kernel a lattice Λ, so $G = T_eG/\Lambda$ is a complex torus.

Exercise 3. Let us study the exponential map $\exp : \mathfrak{su}_2 \to \mathrm{SU}_2 = S^3$

(a) Show that exp is surjective.

(b) The spheres of even and odd radius are sent to the north pole N and the south pole S of $S^3 = \mathrm{SU}(2)$.

(c) Show that outside of the set $\exp^{-1}(\{S, N\})$ the map is a covering.

Exercise 4. Show that the exponential map $\exp : \mathfrak{sl}_2 \to \mathrm{SL}_2(\mathbb{R})$ is not surjective.