SYLLABUS FOR MATH 718: LIE ALGEBRAS

A. OBLOMKOV

Books for the class

Our main source is the book by William Fulton and Joe Harris "Representation Theory: a first course" Other notable book that is useful for the class is James E. Humphreys "Introduction to Lie Algebras and Representation theory". The other sources will be provided by the instructor during the semester.

Course Description

This course offers an introduction to the representation theory of Lie groups and Lie algebras, focusing on the classical examples. We will begin by linearizing the study of Lie groups to their Lie algebras and develop the fundamental tools for analyzing their structure. The core of the course will be a detailed, example-driven exploration of the representations of the classical Lie algebras ($\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$, and $\mathfrak{so}_m(\mathbb{C})$). The final part of the course will briefly touch upon the general classification theory and the Weyl character formula.

Part I: Lie Groups and Lie Algebras

Week 1-2: Foundations. We'll define Lie groups and Lie algebras, explore fundamental examples, and establish the crucial link between them via the **exponential map**.

Week 3-4: The Archetype: Representations of $\mathfrak{sl}_2(\mathbb{C})$. This is the most important example. We will fully classify its irreducible representations using the concepts of **weights** and **highest weight vectors**. We will also explore tensor products of these representations (plethysm).

Week 5-6: The Next Step: Representations of $\mathfrak{sl}_3(\mathbb{C})$. We will generalize the methods from the $\mathfrak{sl}_2\mathbb{C}$ case, introducing the Cartan subalgebra, the root system, and the Weyl group in a concrete setting. We'll classify its representations and see how the complexity grows.

Part II: The Classical Lie Algebras and Their Representations

Week 7: The General Machinery. We'll formalize the tools developed for $\mathfrak{sl}_3(\mathbb{C})$, defining **roots**, **weights**, **Cartan subalgebras**, and the **Weyl group** for an arbitrary semisimple Lie algebra.

Week 8-9: Representations of $\mathfrak{sl}_n(\mathbb{C})$ (Type A). We will systematically describe the irreducible representations of the special linear Lie algebras, connecting them to Young diagrams and Weyl's construction.

Week 10-11: Representations of Symplectic Algebras $\mathfrak{sp}_{2n}(\mathbb{C})$ (Type C). We'll analyze the structure of symplectic Lie algebras and classify their irreducible representations.

Week 12-13: Representations of Orthogonal Algebras $\mathfrak{so}_m(\mathbb{C})$ (Types B & D). We will analyze the structure of orthogonal Lie algebras, noting the differences between the even and odd dimensional cases. We will construct their representations, leading to the discovery of the **spin representations**.

Part III: A Glimpse of General Lie Theory

Week 14: Classification and Characters. A brief overview of the classification of all simple Lie algebras using **Dynkin diagrams**. We will state the **Weyl Character Formula** as a tool for understanding the characters and dimensions of representations. We'll conclude with a brief discussion of real Lie algebras and groups.

Grading

The final grade is based on the weekly homework (0.7) and on the presentation at the end of semester (0.3). The list of topics for the presentation will be offered in class. Suggestions for students on the topics of presentations are also encouraged.

Grade distribution A: [0.8,1]; B: [0.6,0.8); C: [0.4,0.6); D: [0.2,0.4); F: [0,0.2).

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